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ON A PARTICULAR FORM OF STABILIZED CAPILLARY-GRAVITATIONAL WAVES OF FINITE AMPLITUDE AT THE SURFACE OF FLUID OVER AN UNDULATING BED

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The problem of stabilized plane capillary-gravitational waves of finite amplitude at the surface of a stream of perfect incompressible fluid flowing over an undulating bed and subjected to pressure periodically distributed along the surface and defined by some infinite trigonometric series is considered. The intersection of the bed with a vertical plane is assumed to be a periodic curve, called the bed line, defined by some infinite trigonometric series.

The problem is rigorously formulated and reduced to the solution of a system of nonlinear integral and transcendental equations. The solution is constructed in the form of series in powers of a small dimensionless parameter to which amplitudes of the first harmonics of the bed line and of the surface pressure wave are proportional. An approximate equation is derived for the wave profile.

The particular case is considered, when the length of the bed line wave arc is equal to the length of the stabilized free wave line corresponding to the specified flow velocity over a horizontal flat bed and constant pressure along the surface. In such case the parameter of the integral equation is equal to one of the eigenvalues of the kernel of that equation and the solution is constructed in the form of series in powers of the cube root of the small parameter mentioned above.

A similar problem but for constant pressure along the surface was considered by the author in [1, 2] and in his paper presented at the 13-th International Congress on Theoretical and Applied Mechanics (Moscow, 1972 [3]).

Another similar problem of capillary-gravitational waves over an undulating bed was considered in [4], where besides the topological proof of the existence and uniqueness of solution the algorithm for constructing the latter is given, but the calculation of approximations is only outlined and the mechanical meaning of solution is not investigated in depth.

Unlike in [4] the equation of the bed line and the expression for pressure at the surface are specified here in a form which makes it possible to express any approximations in the form of finite sums, and an analysis of the fundamental system of nonlinear integral and transcendental equations by the Liapunov-Schmidt analytical methods and their developments is presented.

1. Statement of problem and derivation of fundamental equations. Let us consider the stabilized plane-parallel motion of a perfect incompressible heavy fluid bounded from above by a free surface subjected to pressure $p_0 = p_0' + p_0(x)$, where $p_0' = \text{const}$ and $p_0(x)$ is a specified periodic function of the horizontal coordinate x. The fluid is bounded from below by an undulating bed whose intersection with the vertical plane of flow is defined by a certain specified periodic doubly differentiable curve L called the bed line. It is assumed that the wavelike line L is symmetric with respect to verticals passing through its crests and the middle of troughs. Let us assume that the specified mean horizontal velocity c of flow is constant for $y = 0$ (see below) and is directed from left to right.

Owing to the periodicity of pressure specified at the stream surface and of the bed line, the free surface is a stationary periodic wave in coordinates attached to the wave progressing at velocity c. Such waves are produced by undulations of the bed and pressure distribution along the fluid surface. If the bed is a horizontal plane and the pressure constant, these waves disappear and the flow is a uniform stream. Waves generated by an undulating bed at varying as well as at constant pressure at the surface will be called induced waves [2] as opposed to free waves which exist in the case of a horizontal flat bed and constant pressure at the surface at certain specific flow velocities.

Let the crest of a wave of curve L lie on some vertical line, and let the unknown wave and the curve representing pressure $p_0(x)$ be symmetric about this vertical and the vertical drawn through the middle of the trough of line L. We superpose the y-axis of the rectangular coordinate system xy on the axis of symmetry of the crest and direct it vertically upward. We locate the coordinate origin O at the point of intersection of the y-axis with line L and direct the x-axis from left to right along the tangent to the bed line. Let the period (or the wave length) of line L be λ. Along the length of the wave between two crests there is at least one trough (in the general case there may be several crests and troughs along this length). It is assumed that line L has horizontal tangents at points $x = 0$ and $x = \pm {}^1/_2\lambda$. We define the angle between a tangent to line L and the x-axis in the form of function $\Theta(s)$, where s is the length of wave arc measured from zero. The positive direction of a tangent is that which corresponds to increasing length of arc s. We denote by $2l$ the length of arc of line L corresponding to the period along x, i.e. for $0 \leqslant x \leqslant \lambda$. At $x = -{}^1/_2\lambda$ and $x = {}^1/_2\lambda$ the lengths of arcs are, respectively, $s = -l$ and $s = l$. Since $\Theta(s)$ is a continuous function of s, which changes its sign at the crest tops and at the middle of troughs, hence

$$\Theta(0) = \Theta(l) = \Theta(-l) = 0 \tag{1.1}$$

By virtue of the specified condition of symmetry we have

$$\Theta(-l + s) = -\Theta(l - s) \tag{1.2}$$

Assuming that the slope of line L is small, in accordance with the condition of periodicity and conditions (1.1) and (1.2), we specify function $\Theta(s)$ in the form of the following trigonometric series:

$$\Theta(s) = \sum_{n=1}^{\infty} \varepsilon^n \beta_n \sin \frac{n\pi s}{l} \tag{1.3}$$

where ε is a small positive dimensionless parameter, β_n are specified real numbers, and the series $\sum_{n=1}^{\infty} \varepsilon^n \beta_n$ is convergent within a circle of radius $\varepsilon_0 > 0$. Using function $\Theta(s)$, we can write the equation for the bed line as

$$x = \int_0^s \cos\Theta(s)\,ds, \qquad y = \int_0^s \sin\Theta(s)\,ds \tag{1.4}$$

which implies that λ — the length of the wave of line L — is defined by formula

$$\lambda = \int_0^{2l} \cos\Theta(s)\,ds \tag{1.5}$$

It follows from (1.3) and (1.5) that λ is the following known function of ε:

$$\lambda = \lambda_0 + \sum_{n=1}^{\infty} \lambda_n \varepsilon^n, \quad \lambda_0 = 2l, \quad \lambda_1 = 0, \quad \lambda_2 = -\frac{\beta_1{}^2 l}{2}, \quad \lambda_3 = 0 \tag{1.6}$$

where λ_n $(n = 4, 5, \ldots)$ are polynomials in β_i. It is assumed that the length of the unknown stabilized wave of the flow over an undulating bed and the period of the specified function $p_0(x)$ are equal λ. We take the flow plane xy as the plane of the complex variable $z = x + iy$. Let φ be the velocity potential, ψ the stream function, $w = \varphi + i\psi$ the complex potential of velocities, and U and V projections of the velocity vector $\mathbf{q}$ on the coordinate axes. We then have

$$dw/dz = -U + iV, \qquad U = -\partial\varphi/\partial x, \qquad V = -\partial\varphi/\partial y$$

To derive fundamental equations of this problem from boundary conditions we conformally map the region occupied by one wave, which is represented by a vertical rectangle bounded from above and below by wavelike curves, into the rectangle

$$0 \leqslant \varphi \leqslant \varphi_0, \qquad 0 \leqslant \psi \leqslant \psi_0$$

in plane w (where $\psi = \psi_0$ is the flow discharge rate per unit of time, $\varphi = 0$ and $\varphi = \varphi_0$ for $x = 0$ and $x = \lambda$), respectively), and then map this rectangle onto the interior of a circular ring whose center is at the zero of plane $u = u_1 + iu_2$. The last transformation is given by formula

$$w = \frac{\varphi_0}{2\pi i} \ln u \tag{1.7}$$

The segment $0 \leqslant \varphi \leqslant \varphi_0$ which corresponds to the free surface now becomes the circumference of the external circle of unit radius, and the segment which corresponds to the bed becomes the circumference of the inner circle of radius $r_0 = \exp(-2\pi\psi_0 / \varphi_0)$ smaller than unity. The ring is slit along segment $(r_0, 1)$. The solution is derived on the assumption that ψ_0 / φ_0 and, consequently, also r_0 are specified and independent of ε (see (1.3)). The image of this ring of plane u in the region of a single wave of plane z is determined by the relationship

$$\frac{dz}{du} = -\frac{\lambda}{2\pi i} \frac{e^{i\omega(u)}}{u}, \qquad \omega(u) = \Phi + i\tau \tag{1.8}$$

Setting $\varphi_0 = c\lambda$, by virtue of (1.7) and (1.8) we obtain

$$dw/dz = -ce^{\tau - i\Phi}$$

This implies that throughout the stream function Φ is equal to the angle between the velocity vector $\mathbf{q}$ and the x-axis, and that

$$q = |\mathbf{q}| = ce^{\tau}$$

Since function $\omega(u)$ is holomorphic, it is represented inside the considered ring of plane u by a Laurent expansion. It can be shown that owing to the symmetry of the wave, bed line and pressure $p_0(x)$, the coefficients of that series must be real. For $u = e^{i\theta}$ (θ is the angle between the radius vector and the u_1-axis) from (1.8) we obtain a differential relationship which, after separation of real and imaginary parts and integration, yields for the wave profile the parametric (1.30)

$$x = -\frac{\lambda}{2\pi}\int_0^\theta e^{-\tau(\eta)} \cos\Phi(\eta)\, d\eta, \qquad y = -\frac{\lambda}{2\pi}\int_0^\theta e^{-\tau(\eta)} \sin\Phi(\eta)\, d\eta \tag{1.9}$$

In determining y we assume that the coordinate origin is transferred to the top of the wave crest; and in (1.9)

$$\tau(\eta) = \tau(1, \eta), \qquad \Phi(\eta) = \Phi(1, \eta)$$

It follows from (1.9) that for solving the problem it is necessary to determine in addition to $\Phi(\theta)$ also $\tau(\theta)$. Owing to the symmetry of the unknown wave about the vertical passing through the crest, function $\tau(\theta)$ is even and function $\Phi(\theta)$ is odd. Hence they can be represented by the following trigonometric series:

$$-\tau(\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta, \qquad \Phi(\theta) = \sum_{n=1}^{\infty} B_n \sin n\theta \tag{1.10}$$

It is shown in the theory of analytic functions that along the external circumference the relationships

$$-\tau(\theta) - A_0 = \int_0^{2\pi} K(\eta, \theta) \frac{d\Phi}{d\eta}\, d\eta - 2\int_0^{2\pi} N(\eta, \theta) \frac{d\Phi^*}{d\eta}\, d\eta \tag{1.11}$$

$$K(\eta, \theta) = \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\cos n\eta \cos n\theta}{\nu_n'}, \qquad N(\eta, \theta) = \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\cos n\eta \cos n\theta}{\nu_n^*}.$$

$$\nu_n' = n\frac{r_0^{-n} - r_0^{n}}{r_0^{-n} + r_0^{n}}, \qquad \nu_n^* = n(r_0^{-n} - r_0^{n}), \qquad \frac{4}{\nu_n^{*2}} = \frac{1}{\nu_n'^2} - \frac{1}{n^2}$$

$$\Phi(\theta) = \int_0^{2\pi} K_0(\eta, \theta) \frac{d\tau}{d\eta}\, d\eta + 2\int_0^{2\pi} M(\eta, \theta) \frac{d\Phi^*}{d\eta}\, d\eta$$

$$K_0(\eta, \theta) = \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\sin n\eta \sin n\theta}{\nu_n^*}, \qquad M(\eta, \theta) = \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\cos n\eta \sin n\theta}{\nu_n^{**}}$$

$$\nu_n'' = n\frac{r_0^{-n} + r_0^{n}}{r_0^{-n} - r_0^{n}}, \qquad \nu_n^{**} = n(r_0^{n} + r_0^{-n}), \qquad \nu_n'\nu_n'' = n^2$$

where $\tau^*(\theta) = \tau(r_0, \theta)$ and $\Phi^*(\theta) = \Phi(r_0, \theta)$ which follow from Villat's equations for the ring. Owing to the symmetry of the bed line, expansions (1.10) are valid for these functions but with other A_n and B_n ($n = 1, 2, 3, \ldots$). Passing to the boundary condition at the surface, we use for it the Bernoulli integral

$$p / \rho = C - gy - {}^1\!/_2 q^2 \tag{1.12}$$

where C is a constant, g is the acceleration of gravity, and ρ is the density. The pressure difference at the free surface is balanced by the normal component of surface tension forces. In accordance with the law of Laplace for these forces we have

$$p - p_0 = \pm \mu / R \tag{1.13}$$

where p is the pressure from the fluid side, $p_0 = p_0' + p_0(x)$ is the pressure from the side of the free surface, μ is the capillary constant, and R is the radius of curvature at points of the surface. From this, by expressing the curvature in terms of $d\Phi / d\theta$, we obtain

$$p - p_0 = \frac{2\pi\mu}{\lambda c} q \frac{d\Phi}{d\theta} \tag{1.14}$$

Substituting the expression for p from (1.14) into (1.12), we have

$$\frac{d\Phi}{d\theta} = \nu \left[\delta e^{-\tau} - e^{\tau} - \frac{2\pi}{\lambda} \varkappa y e^{-\tau} - p_0{}^*(x) e^{-\tau}\right] \tag{1.15}$$

$$\nu = \frac{\lambda c^2 \rho}{4\pi\mu}, \quad \delta = \frac{2(C\rho - p_0')}{\rho c^2}, \quad \varkappa = \frac{g\lambda}{\pi c^2}, \quad p_0{}^*(x) = \frac{2p_0(x)}{\rho c^2} \tag{1.16}$$

where x and y are determined by formulas (1.9) as functions of θ. Separating in the right-hand side of (1.15) terms linear with respect to Φ and τ, we obtain

$$\frac{d\Phi}{d\theta} = \nu \Big\{\delta - 1 - (\delta + 1)\tau + \varkappa \int_0^\theta \Phi(\eta)\, d\eta - \tag{1.17}$$

$$S(\theta)(1 - \tau) + F[\tau, \Phi, S, \delta]\Big\}$$

$$F[\tau, \Phi, S, \delta] = \delta(e^{-\tau} - 1 + \tau) - (e^{\tau} - 1 - \tau) +$$

$$\varkappa e^{-\tau} \int_0^\theta [e^{-\tau(\eta)} \sin\Phi(\eta) - \Phi(\eta)]\, d\eta -$$

$$\varkappa \int_0^\theta \Phi(\eta)\, d\eta] + \varkappa e^{-\tau} \int_0^\theta \Phi(\eta)\, d\eta - S(\theta)(e^{-\tau} - 1 + \tau)$$

We assume here that within the accuracy of the constant included in p_0'

$$p_0^*(x) = \sum_{n=1}^{\infty} \varepsilon^n d_n \cos\frac{2\pi n}{\lambda} x = S(\theta) \tag{1.18}$$

where ε is the same small parameter which appears in (1.3), d_n are specified real numbers, and the expansion $\Sigma \varepsilon^n d_n$ is convergent in a circle of radius $\varepsilon_0 > 0$ (see (1.3)). To determine $S(\theta)$ we must substitute into (1.18) the values of x / λ obtained from equation

$$\frac{x}{\lambda} = -\frac{1}{2\pi} \int_0^\theta e^{-\tau(\eta)} \cos\Phi(\eta)\, d\eta \tag{1.19}$$

which follows from (1.9). The expression for y is taken into account in (1.17).

Let us determine more accurately the parameters in Eq. (1.17). It follows from (1.6) and (1.10) that

$$\nu = \nu^{(0)} + \sum_{n=1}^{\infty} \nu^{(n)} \varepsilon^n, \quad \nu^{(0)} = \frac{c^2 \rho \lambda_0}{4\pi\mu}, \quad \nu^{(n)} = \frac{\nu^{(0)}}{\lambda_0} \lambda_n \tag{1.20}$$

$$\varkappa = \varkappa_0 + \sum_{n=1}^{\infty} \varkappa_n \varepsilon^n, \quad \varkappa_0 = \frac{g\lambda_0}{\pi c^2}, \quad \varkappa_n = \frac{\varkappa_0}{\lambda_0} \lambda_n$$

By virtue of (1.20) Eq. (1.17) assumes the form

$$\frac{d\Phi}{d\theta} = \nu^{(0)}\Big\{\delta - 1 - (\delta + 1)\tau + \varkappa_0 \int_0^\theta \Phi(\eta)\,d\eta + \sum_{n=1}^{\infty} \varepsilon^n \varkappa_n \int_0^\theta \Phi(\eta)\,d\eta - \tag{1.21}$$

$$S(\theta)(1-\tau) + F[\tau, \Phi, S, \delta]\Big\} + \sum_{n=1}^{\infty} \varepsilon^n \nu^{(n)} \Big\{\ldots\Big\}$$

where the expression omitted in the second braces is the same as in the first ones. Terms linear with respect to τ, Φ and ε in braces in (1.21) are transformed by using formulas (1.11) and integration by parts. After this we combine in the first braces terms (with coefficients 2 and $-\varkappa_0$) with the same integrand $d\Phi\,/\,d\eta$ and different kernels $K(\eta, \theta)$ and $K_2(\eta, \theta)$ from (1.23). Since velocity c is specified, hence parameters $\nu^{(0)}$ and $\varkappa_0$ are fixed, and δ is determined by the condition of periodicity: $\Phi(\theta + 2\pi) = \Phi(\theta)$. Since the right-hand side of Eq. (1.21) contains parameter ε, hence the solution and, consequently also δ depend on ε. Let us set

$$\delta = \delta_0 + \delta'(\varepsilon) \tag{1.22}$$

For $\varepsilon \to 0$ from the condition of periodicity we find that $\delta_0 = 1$ since the solution and $\delta'(\varepsilon)$ then tend to vanish.

After all these transformations with allowance for (1.22), Eq. (1.21) assumes its final form (dots in the second braces stand for the last six terms appearing in the first braces)

$$\zeta(\theta) = \nu^{(0)}\Big\{\int_0^{2\pi} K^*(\eta, \theta)\,\zeta(\eta)\,d\eta + \delta'(\varepsilon) - \tag{1.23}$$

$$2(2 + \delta'(\varepsilon)) \int_0^{2\pi} N(\eta, \theta)\,\zeta^*(\eta)\,d\eta + (2 + \delta'(\varepsilon))\,A_0(\varepsilon) +$$

$$\delta'(\varepsilon) \int_0^{2\pi} K(\eta, \theta)\,\zeta(\eta)\,d\eta + \varkappa_0 \int_0^{2\pi} K_2(\eta, 0)\,\zeta(\eta)\,d\eta + \Psi(\theta, \varepsilon)\Big\} +$$

$$\sum_{n=1}^{\infty} \nu^{(n)} \varepsilon^n \Big\{2 \int_0^{2\pi} K(\eta, \theta)\,\zeta(\eta)\,d\eta - \varkappa_0 \int_0^{2\pi} K_2(\eta, \theta)\,\zeta(\eta)\,d\eta + \ldots\Big\}$$

$$\zeta(\theta) = d\Phi\,/\,d\theta, \qquad \zeta^*(\theta) = d\Phi^*\,/\,d\theta$$

$$\Psi(\theta, \varepsilon) = \sum_{n=1}^{\infty} \varkappa_n \varepsilon^n \int_0^\theta \Phi(\eta)\,d\eta - S(\theta)\Big[1 + A_0 + \int_0^{2\pi} K(\eta, \theta)\,\zeta(\eta)\,d\eta -$$

$$2 \int_0^{2\pi} N(\eta, \theta)\,\zeta^*(\eta)\,d\eta\Big] + F[\tau, \Phi, S, 1 + \delta'(\varepsilon)]$$

$$K_2(\eta, \theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\eta \cos n\theta}{n^2}, \quad K^*(\eta, \theta) = \sum_{n=1}^{\infty} \frac{\varphi_n(\eta)\,\varphi_n(\theta)}{\nu_n}$$

$$\nu_n = \frac{n^2}{2\nu_n'' - \varkappa_0}, \quad \varphi_n(\theta) = \frac{\cos n\theta}{\sqrt{\pi}}$$

where ν_n are eigenvalues and $\varphi_n(\theta)$ eigenfunctions of kernel $K^*(\eta, \theta)$. The conditions of periodicity for function $\Phi(\theta)$ yields the expression

$$\delta'(\varepsilon) = -\varkappa_0 \int_0^{2\pi} K_2(\eta, 0)\zeta(\eta)\,d\eta - (2+\delta'(\varepsilon))A_0 - \tag{1.24}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \Psi^{\cdot}(\theta,\varepsilon)\,d\theta - \frac{1}{2\pi\nu^{(0)}}\sum_{n=1}^{\infty} \nu^{(n)}\varepsilon^n \Big\{\Big[\delta'(\varepsilon) + (2+\delta'(\varepsilon))A_0 +$$

$$\varkappa_0 \int_0^{2\pi} K_2(\eta,0)\zeta(\eta)\,d\eta\Big]2\pi + \int_0^{2\pi} \Psi^{\cdot}(\theta,\varepsilon)\,d\theta\Big\}$$

Let us consider boundary condition at the bed for $r = r_0$. The flow must evidently follow the contour of the bed. Because of (1.3) the condition for this is of the form

$$\Phi^*(\theta) = \Theta[s(\theta)] = \sum_{n=1}^{\infty} \varepsilon^n \beta_n \sin\frac{n\pi s(\theta)}{l} \tag{1.25}$$

To obtain this condition in its final form it is necessary to determine function $s(\theta)$. We recall that for $r = r_0$

$$dz = -\frac{\lambda}{2\pi} e^{-\tau^*(\theta)+i\Phi^*(\theta)}\,d\theta$$

Hence

$$ds = |dz| = -\frac{\lambda}{2\pi} e^{-\tau^*(\theta)}\,d\theta \tag{1.26}$$

The minus sign in this formula shows that positive increments of arc s correspond to decrements of θ. From (1.26) we have

$$s(\theta) = -\frac{\lambda}{2\pi}\int_0^{\theta} e^{-\tau^*(\eta)}\,d\eta \tag{1.27}$$

We select coefficient $A_0(\varepsilon)$ so as to have the length of the bed line arc, which corresponds to a period, equal to the specified value of $2l$. In accordance with (1.27) this condition is determined by formula

$$2l = -\frac{\lambda}{2\pi}\int_0^{-2\pi} e^{-\tau^*(\eta)}\,d\eta$$

or

$$2le^{-A_0(\varepsilon)} = \frac{\lambda}{2\pi}\int_0^{2\pi} e^{-\tau^*(-\eta)-A_0(\varepsilon)}\,d\eta \tag{1.28}$$

and by virtue of (1.10) for τ^* and Φ^* the exponent $-\tau^*(-\eta) - A_0(\varepsilon)$ does not contain $A_0(\varepsilon)$.

Note that another equation may be derived for defining the equality of bed line and surface wave lengths of the fluid. Using the relationship between $\nu_n{}^{*2}$ and $\nu_n{}'^2$ it would be possible to show that this condition which follows from (1.23) is equivalent to (1.28).

It follows from (1.28) that the expansion of $s(\theta)$ in powers of ε has a single secular term, hence

$$s(\theta) = -\frac{l}{\pi}\theta + \sum_{n=1}^{\infty}\varepsilon^n s_n(\theta) = -\frac{l}{\pi}\theta + s'(\theta) \tag{1.29}$$

Differentiating (1.25) with respect to θ and allowing for (1.27), we represent the boundary condition at the bed in the form

$$\zeta^*(\theta) = -\frac{d\Theta(s)}{ds}\frac{\lambda}{2\pi}e^{-\tau^*(\theta)} \tag{1.30}$$

Using expansion (1.29) it would be possible to obtain (1.30) in the form given in [2].

Function $\tau^*(\theta)$ on Eqs. (1.27), (1.28) and (1.30) is determined by formula

$$-\tau^*(\theta) - A_0 = -\int_0^{2\pi} K(\eta, \theta) \frac{d\Phi^*}{d\eta} d\eta + 2\int_0^{2\pi} N(\eta, \theta) \frac{d\Phi}{d\eta} d\eta \qquad (1.31)$$

which is derived similarly to the first of formulas (1.11).

The problem is thus reduced to the determination of functions

$$\zeta(\theta, \varepsilon) = d\Phi / d\theta, \quad \zeta^*(\theta, \varepsilon) = d\Phi^* / d\theta, \quad x(\theta, \varepsilon) / \lambda, \quad s(\theta, \varepsilon)$$

and constants $\delta = 1 + \delta'(\varepsilon)$ and $A_0(\varepsilon)$ by the system of Eqs. (1.19), (1.23), (1.24), (1.27), (1.28) and (1.30), with $\tau(\theta, \varepsilon)$ and $\tau^*(\theta, \varepsilon)$ determined by (1.11) and (1.31) and

$$\Phi(\theta, \varepsilon) = \int_0 \zeta(\eta, \varepsilon)\, d\eta, \quad \Phi^*(\theta, \varepsilon) = \int_0^\theta \zeta^*(\eta, \varepsilon)\, d\eta \qquad (1.32)$$

Eliminating with the use of formulas (1.11), (1.19), (1.27), (1.28) and (1.31) from this system $\tau(\theta)$, $\tau^*(\theta)$, $x(\theta)/\lambda$, $s(\theta)$ and $A_0(\varepsilon)$ and representing $\Phi(\theta, \varepsilon)$ and $\Phi^*(\theta, \varepsilon)$ in the form (1.32), we find that (1.23) and (1.30) are nonlinear integral equations with respect to $\zeta(\theta, \varepsilon)$ and $\zeta^*(\theta, \varepsilon)$, and (1.24) is a transcendental equation with respect to $\delta'(\varepsilon)$ with functionals relative to the unknown functions. For the convenience of solution it is, however, expedient not to carry out this elimination and consider only the integral equation (1.23) as nonlinear with respect to $\zeta(\theta, \varepsilon)$, while considering the remaining equations, including (1.23) as nonlinear transcendental equations with respect to functions $\zeta^*(\theta, \varepsilon)$, $x(\theta, \varepsilon)/\lambda$ and $s(\theta, \varepsilon)$ and constants $\delta'(\varepsilon)$ and $A_0(\varepsilon)$ with linear operators and functionals relative to the unknown functions.

In solving this problem we have to consider two cases: (1) $\nu^{(0)} \neq \nu_n$ and (2) $\nu^{(0)} = \nu_n$.

In the first case the solution is constructed in the form of series expansions in integral powers of parameter ε, and in the second in $\varepsilon^{1/3}$. In both cases Fredholm linear integral equations of the second kind with kernel $K^*(\eta, \theta)$ and parameter $\nu^{(0)}$ are obtained for the coefficients of expansion $\zeta(\theta, \varepsilon)$. For the coefficients of expansions of remaining unknowns, a system of linear algebraic equations which are always solvable, is obtained. Equations for the first coefficients of these expansions for $\nu^{(0)} = \nu_n$ and $\nu^{(0)} \neq \nu_n$ are analyzed in Sect. 2.

We note the mechanical meaning of the limit solution for $\varepsilon \to 0$. We can show, as in [2], that at the limit the flow becomes a uniform stream over a horizontal bed with a horizontal free surface.

2. Solution of the linear problem. 2.1. Solution of the linear problem for $\nu^{(0)} = \nu_n$ and analysis of the kernel of Eq. (1.23). The solutions of Eqs. (1.23) and (1.24) in the form of series in $\varepsilon^{1/3}$ are

$$\zeta_1(\theta) = \nu^{(0)}\left[\int_0^{2\pi} K^*(\eta, \theta)\, \zeta_1(\eta)\, d\eta + \delta_1 + 2A_{01} + \varkappa_0 \int_0^{2\pi} K_2(\eta, 0)\, \zeta_1(\eta)\, d\eta\right] \qquad (2.1)$$

$$\delta_1 = -\varkappa_0 \int_0^{2\pi} K_2(\eta, 0)\, \zeta_1(\eta)\, d\eta - 2A_{01} \qquad (2.2)$$

We obtain the same system if we assume that $\zeta^*(\theta) \equiv 0$ and $S(\theta) \equiv 0$ in (1.23) and (1.24), as in the case of a free wave over a flat bed, and restrict the analysis to linear terms.

Eliminating δ_1 from (2.2) and (2.1) and discarding the subscript, we obtain

$$\zeta(\theta) = \nu^{(0)} \int_0^{2\pi} K^*(\eta, \theta)\zeta(\eta)\,d\eta \tag{2.3}$$

This equation is a Fredholm homogeneous linear equation of the second kind, hence by the second Fredholm theorem it has a nonzero solution for $\nu^{(0)} = \nu_n$, where ν_n is the eigenvalue of kernel $K^*(\eta, \theta)$. On the other hand, by virtue of (1.16) parameter $\nu^{(0)} > 0$ and in accordance with (1.23) ν_n depends on n and $\varkappa_0$, with the latter considered fixed. We must, therefore, investigate the dependence of ν_n on n for fixed $\varkappa_0$. A detailed investigation of this dependence is given in [6].

Let us now consider n as fixed and investigate the relationship between $\nu^{(0)}$ and $\varkappa_0$ for which a nonzero solution of Eq. (2.3) exists. Setting $\nu^{(0)} = \nu_n$, from (1.23) we obtain

$$\frac{1}{\nu^{(0)}} = \frac{1}{n^2}(2\nu_n'' - \varkappa_0) \tag{2.4}$$

Substituting into this equation the expressions for $\nu^{(0)}$ and $\varkappa_0$ from (1.16) we obtain the known dependence between c^2 and λ_0

$$c^2 = \left(\frac{2\pi\mu n}{\lambda_0\rho} + \frac{g\lambda_0}{2\pi n}\right) \operatorname{th}\left(2\pi n\frac{h}{\lambda_0}\right) \tag{2.5}$$

Formulas (2.4) and (2.5) were also analyzed in detail in [6].

Here we present only some of the results of the analysis of the linear problem solution.

Theorem 2.1. Let

$$\frac{1}{\nu^{(0)}} = \frac{1}{n^2}(2\nu_n'' - \varkappa_0)$$

where n is a fixed positive integer. Then for all $\varkappa_0$ in the interval $0 < \varkappa_0 < 2\nu_n''$ Eq. (2.3) has the nontrivial solution

$$\zeta(\theta) = C_1\varphi_n(\theta) = \frac{C_1}{\sqrt{\pi}}\cos n\theta$$

If

$$\varkappa_0 = \varkappa_0^{(m)} = \frac{2(m^2\nu_n'' - n^2\nu_m'')}{m^2 - n^2}$$

where m is a positive integer, then

$$\zeta(\theta) = C_2\varphi_m(\theta) = \frac{C_2}{\sqrt{\pi}}\cos m\theta$$

are particular solutions linearly independent of $\varphi_n(\theta)$ and the general solution is

$$\zeta(\theta) = C_1\varphi_n(\theta) + C_2\varphi_m(\theta) = \frac{C_1}{\sqrt{\pi}}\cos n\theta + \frac{C_2}{\sqrt{\pi}}\cos m\theta$$

The value $\varkappa_0 = \varkappa_0^{(m)}$ is called bifurcational, and waves corresponding to that value and determined by the solution in the form of a sum of two harmonics are called double-waves. The eigenvalue $\nu_n = \nu_m$ related to $\varkappa_0 = \varkappa_0^{(m)}$ is binary.

Theorem 2.2. Curve $c^2 = c^2(\lambda_0)$ representing Eq. (2.5) has a vertical asymptote $\lambda_0 = 0$ and a horizontal one $c^2 = gh$. The value $c^2_{\min}$ which corresponds to $\lambda_0 = \lambda_0^*$, where λ_0^* is the positive root of some transcendental equation. We call the related

$\varkappa_0 = \varkappa_0^*$ critical and from (1.16) we have for it the expression

$$\varkappa_0^* = \frac{g\lambda_0^*}{\pi c_{\min}^2}$$

The branch of curve $c^2 = c^2(\lambda_0)$ which corresponds to $0 < \lambda_0 < \lambda_0^*$ or $0 < \varkappa_0 < \varkappa_0^*$ defines waves called capillary-gravitational, while waves which occur for $\lambda_0 > \lambda_0^*$ or $\varkappa_0^* < \varkappa_0 < 2\nu_n''$ are called gravitational-capillary.

2.2. Solution of the linear problem in the case of $\nu^{(0)} \neq \nu_n$. To analyze in a linear approximation the possible forms of the free surface in dependence on the wave propagation velocity we assume that the bed line is specified by

$$\Theta(s) = \varepsilon \sum_{n=1}^{\infty} \beta_n \cos \frac{n\pi s}{l} \tag{2.6}$$

Then

$$\zeta^*(\theta, \varepsilon) = \varepsilon\zeta_1^*(\theta) = -\varepsilon \sum_{i=1}^{\infty} \beta_i \cos i\theta \tag{2.7}$$

As the result of solving the related linear nonhomogeneous integral equation for $\nu^{(0)} \neq \nu_n$ we obtain for function $\zeta(\theta, \varepsilon)$ the following expression:

$$\zeta(\theta, \varepsilon) = \varepsilon\nu^{(0)} \sum_{i=1}^{\infty} \frac{\nu_i}{(\nu_i - \nu^{(0)})} \left(\frac{4\beta_i}{\nu_i^*} - d_i\right) \cos i\theta \tag{2.8}$$

Integrating (2.7) and (2.8), we obtain

$$\Phi^*(\theta, \varepsilon) = -\varepsilon \sum_{i=1}^{\infty} \frac{\beta_i}{i} \sin i\theta \tag{2.9}$$

$$\Phi(\theta, \varepsilon) = \varepsilon\nu^{(0)} \sum_{i=1}^{\infty} \frac{\nu_i}{i(\nu_i - \nu^{(0)})} \left(\frac{4\beta_i}{\nu_i^*} - d_i\right) \sin i\theta \tag{2.10}$$

It can be shown that

$$\begin{aligned} \nu_i - \nu^{(0)} > 0 \quad \text{for} \quad c < c_i \\ \nu_i - \nu^{(0)} < 0 \quad \text{for} \quad c > c_i \end{aligned} \tag{2.11}$$

where c_i denotes the wave velocity determined by formula (2.5) for $i = n$. The condition that

$$c^2 > \frac{g\lambda_0}{2\pi n} \operatorname{th}(2\pi n\psi_0 / \varphi_0) \tag{2.12}$$

which follows from the positiveness of ν_n is taken into consideration in the proof. Since $\nu^{(0)} > 0$ and $\nu_i > 0$, the signs of coefficients in formula (2.10) are determined by the signs of

$$\frac{1}{\nu_i - \nu^{(0)}} \left(\frac{4\beta_i}{\nu_i^*} - d_i\right)$$

Let $(4\beta_i/\nu_i^* - d_i) > 0$ and $\beta_i > 0$, then it follows from (2.11) that the coefficients at related terms in (2.9) and (2.10) hve different signs for $c < c_i$ and the same signs for $c > c_i$. The analysis of formulas (2.9) and (2.10) with the use of inequality (2.11) leads to the following conclusions.

Let the inequality $c_{2n-1} < c < c_{2n}$ be satisfied for velocity c. Then for c close to c_{2n-1} the principal term of expansion (2.10) has a minus sign. If, however, c is close to c_{2n}, the sign of the principal term in that equation is positive. In the first case crests and troughs of the free surface lie over the crests and troughs of the bed line. In the second case the troughs of surface waves lie over the crests of the bed line and the crests of the former lie over the troughs of the latter. The relative position of crests and troughs of the bed line and of the surface wave profile in the next following interval $c_{2n} < c < c_{2n+1}$ is similar. It was assumed in this analysis that all $\beta_i > 0$ and that the first term in formula (2.9) is the principal term of the expansion which defines the shape of the bed line.

Analysis of the solution of the linear problem for $\nu^{(0)} \neq \nu_n$ and the bed line specified in the form of series (1.3) is given in Sect. 4 and Note 4.3 (see also p. 380 in [1]).

3. Solution of fundamental equations of the problem. As was already noted at the end of Sect. 1, it is necessary to consider the two cases of $\nu^{(0)} \neq \nu_n$ and $\nu^{(0)} = \nu_n$ when solving the system of Eqs. (1.19), (1.23), (1.24), (1.27), (1.28) and (1.30). We shall indicate the method of solution derivation in both cases. In the first case we present the results of determination of the first three approximations. In the second case a detailed analysis of $\nu^{(0)} = \nu_1$ is given as an example. Parameter $\varkappa_0$ is selected so as to have the eigenvalue of ν_1 simple and positive. The first two approximations are completely calculated, while the third approximation is not entirely determined. For $\nu^{(0)} = \nu_n = \nu_m$ $(n \neq m)$ we present only the method of constructing the solution.

3.1. The case of $\nu^{(0)} \neq \nu_n$. As previously noted, the solution in this case is constructed in terms of expansions in integral powers of ε. For every coefficient of expansion of function $\zeta(\theta, \varepsilon)$ a linear nonhomogeneous Fredholm integral equation of the second kind with kernel $K^*(\eta, \theta)$ and parameter $\nu^{(0)}$ is obtained. All these equations are successively solved by the first Fredholm theory, and for the determination of the coefficients of expansions of remaining unknowns we obtain a system of linear algebraic equations. In this always solvable system the coefficients of a particular approximation are explicitly expressed in terms of quantities determined in preceding approximations.

The expressions for $\zeta^*(\theta, \varepsilon)$, $\zeta(\theta, \varepsilon)$, $\delta'(\varepsilon)$ and $A_0(\varepsilon)$, determined by the first three approximations are

$$\zeta^*(\theta, \varepsilon) = -\varepsilon\beta_1 \cos\theta - \varepsilon^2 D_{22} \cos 2\theta - \varepsilon^3 (D_{13}\cos\theta + D_{33}\cos 3\theta) \quad (3.1)$$

$$\zeta(\theta, \varepsilon) = \varepsilon C_{11} \cos\theta + \varepsilon^2 C_{22} \cos 2\theta + \varepsilon^3 (C_{13}\cos\theta + C_{33}\cos 3\theta)$$

$$\delta'(\varepsilon) = -\varepsilon\varkappa_0 C_{11} - \varepsilon^2({}^1/_4\varkappa_0 C_{22} + 2A_{02} + {}^1/_4\varkappa_0 C_{11}E_{11}) + \varepsilon^3\delta_3'$$

$$A_0(\varepsilon) = \varepsilon^2 A_{02} = -\frac{1}{4}\varepsilon^2\left[\left(\frac{\beta_1}{\nu_1'} + \frac{2}{\nu_1^*}C_{11}\right) - \beta_1^2\right]$$

where

$$C_{11} = \frac{\nu^{(0)}\nu_1}{\nu_1 - \nu^{(0)}}\left(\frac{4\beta_1}{\nu_1^*} - d_1\right) \quad (3.2)$$

$$C_{22} = \frac{\nu^{(0)}\nu_2}{\nu_2 - \nu^{(0)}}\left[\frac{4}{\nu_2^*}D_{22} + \left(d_1 + \frac{3}{4}\varkappa_0 C_{11}\right)E_{11} - d_2\right]$$

$$D_{22} = \beta_1\left(\frac{\beta_1}{\nu_1'} + \frac{2C_{11}}{\nu_1^*}\right) + 2\beta_2, \quad E_{11} = -\left(\frac{1}{\nu_1'}C_{11} + \frac{2\beta_1}{\nu_1^*}\right)$$

$$E_{22} = -\left(\frac{1}{\nu_2'}C_{22} + \frac{2}{\nu_2^*}D_{22}\right), \quad C_{13} = \frac{\nu^{(0)}\nu_1}{\nu_1 - \nu^{(0)}}C_{13}^*, \quad C_{33} = \frac{\nu^{(0)}\nu_3}{\nu_3 - \nu^{(0)}}C_{33}^*$$

Here C_{13}^* is a linear function of C_{11}^3, $C_{11}^2\beta_1$, $C_{11}\beta_1^2$, $C_{11}C_{22}$, $C_{22}\beta_1$, β_1^3, $C_{11}\beta_2$, $\beta_1\beta_2$, $C_{11}d_2$, $C_{11}d_1\beta_1$, $C_{11}^2d_1$, d_1C_{22}, β_1d_2, $\beta_1^2d_1$; D_{13} does not explicitly depend on d_1 and d_2 and is a linear function of the same products of coefficients C_{11}, C_{22}, β_1 and β_2, as C_{13}^*, except C_{11}^3; C_{33}^* is a linear function of the same arguments as C_{13}^* and, in addition, of β_3 and d_3; D_{33} does not explicitly depend on d_1, d_2 and d_3 and is a linear function of β_3 and of the same products of coefficients C_{11}, C_{22}, β_1, β_2 as C_{33}^*, except C_{11}^3; δ_3' is a linear function of C_{13}, C_{33}, C_{11}^3, $C_{11}^2\beta_1$, $\beta_1^2C_{11}$, $C_{11}E_{11}^2$, $C_{11}E_{22}$, $E_{11}C_{22}$.

3.2. The case of $\nu^{(0)} = \nu_1$. In this case, when the solution is derived in the form of series in $\varepsilon^{1/3}$ for the first coefficient of expansion $\zeta(\theta, \varepsilon)$ we obtain a homogeneous linear integral Fredholm equation of the second kind with $\nu^{(0)} = \nu_1$, which is solved by the second Fredholm theorem. Equations for all subsequent coefficients are the same but nonhomogeneous for the same parameter $\nu^{(0)} = \nu_1$. These equations are solved by the third Fredholm theorem. The coefficient in the nth approximation of the solution of the homogeneous equation is determined by the condition of solvability of the equation in the $(n+2)$-nd approximation.

Each of the coefficients C_{11}, C_{12} and C_{13} are successively determined by the related condition of solvability of the equation in the third, fourth and fifth approximations. Coefficient C_{13} was not calculated, because the fifth approximation was not determined. The coefficients of expansions of remaining unknowns are determined as in the case of $\nu^{(0)} \neq \nu_n$.

The expressions for $\zeta(\theta, \varepsilon)$, $\zeta^*(\theta, \varepsilon)$, $\delta'(\varepsilon)$ and $A_0(\varepsilon)$, determined by the first three approximations are

$$\zeta^*(\theta,\varepsilon) = -\varepsilon\beta_1\cos\theta \tag{3.3}$$

$$\zeta(\theta,\varepsilon) = \varepsilon^{1/3}C_{11}\cos\theta + \varepsilon^{2/3}C_{22}\cos 2\theta + \varepsilon(C_{13}\cos\theta + C_{33}\cos 3\theta)$$

$$\delta'(\varepsilon) = -\varepsilon^{1/3}\varkappa_0C_{11} + \varepsilon^{2/3}\left[C_{11}^2\left(\frac{2}{\nu_1^{*2}} + \frac{\varkappa_0}{4\nu_1'}\right) - \frac{1}{4}\varkappa_0C_{22}\right] + \varepsilon\delta_3'$$

$$A_0(\varepsilon) = -\frac{1}{4}\varepsilon^{2/3}\left(\frac{1}{\nu_1'^2} - 1\right)C_{11}^2$$

where

$$C_{11} = \left(d_1 - \frac{4\beta_1}{\nu_1^*}\right)^{1/3}\alpha^{1/3} \tag{3.4}$$

$$\alpha = \frac{32\nu_1'^3(\nu_2-\nu_1)}{(\nu_2-\nu_1)[8(3-2\nu_1'^2) + 12\varkappa_0\nu_1'(1-\nu_1'^2)] + 9\varkappa_0^2\nu_1\nu_2\nu_1'}$$

$$C_{22} = -\frac{3}{4}\frac{\nu_1\nu_2\varkappa_0}{\nu_1'(\nu_2-\nu_1)}C_{11}^2, \quad C_{33} = \frac{\nu_1\nu_3}{\nu_3-\nu_1}C_{33}^*$$

where C_{33}^* is a linear function of C_{11}^3 and $C_{11}C_{22}$; δ_3' is a linear function of C_{13}, C_{33}, C_{11}^3 and $C_{11}C_{22}$, and coefficient $C_{12} = 0$.

We recall that in both cases $\tau(\theta, \varepsilon)$ and $\tau^*(\theta, \varepsilon)$ are determined by (1.11) and (1.31), and $\Phi(\theta, \varepsilon)$ and $\Phi^*(\theta, \varepsilon)$ by (1.32).

3.3. The case of $\nu^{(0)} = \nu_n = \nu_m$ $(n \neq m)$. In this case the solution is derived similarly to the case of $\nu^{(0)} = \nu_n$, but the solution of the homogeneous integral equation contains the sum $C_{in}\cos n\theta + C_{im}\cos m\theta$ in every ith approximation. In the general case coefficients C_{in} and C_{im} are determined by the condition of solvability of the equation in the $(i+2)$-nd approximation.

4. Determination of the wave profile. The wave profile is determined in the parametric form $x(\theta, \varepsilon)$ and $y(\theta, \varepsilon)$ by formulas (1.9). We pass to dimensionless coordinates x/λ and y/λ without altering the notation and, after substituting the obtained $\Phi(\theta,\varepsilon)$ and $\tau(\theta,\varepsilon)$, we obtain the parametric equations for the profile. Eliminating from these equations θ, we reduce the equation for the profile to the form $y = y(x, \varepsilon)$.

Assuming $2\pi = k$, the equation of the wave profile approximate to within third order terms in both cases are:

In the case of $\nu^{(0)} \neq \nu_n$

$$y(x, \varepsilon) = \frac{1}{k}\Big\{\varepsilon C_{11}(\cos kx - 1) + \frac{1}{4}\varepsilon^2 (C_{22} - E_{11}C_{11})(\cos 2kx - 1) + \frac{1}{6}\varepsilon^3\Big[6C_{13} + \frac{3}{8\nu_1'^2}(3\nu_1'^2 - 4)C_{11}^3 - 6\frac{\beta_1}{\nu_1'\nu_1^*}C_{11}\Big(C_{11} - \frac{\beta_1}{\nu_1'\nu_1^*}\Big) + \frac{3}{8}C_{11}E_{11}^2 - \frac{3}{2}C_{11}E_{22} + 3C_{22}E_{11}\Big](\cos kx - 1) + \frac{1}{6}\varepsilon^3\Big[\frac{2}{3}C_{33} - \frac{7}{24}C_{11}^3 + \frac{5}{8}C_{11}E_{11}^2 - \frac{1}{2}C_{11}E_{22} - C_{22}E_{11}\Big](\cos 3kx - 1) \tag{4.1}$$

where coefficients C_{ij} and E_{ij} are defined by formulas (3.2).

In the case of $\nu^{(0)} = \nu_1$ the expression for $y(x, \varepsilon)$ is obtained from (4.1) by substituting in it $\varepsilon^{1/3}$ for ε, and in the expressions for E_{11}, E_{22} and D_{22} setting $\beta_1 = \beta_2 = 0$ and $d_1 = d_2 = 0$. In this case coefficients C_{ij} are determined by formulas (3.4).

Note 4.1 When determining $y(x, \varepsilon)$ we transfer the coordinate origin of the point of intersection of the Oy-axis with the wave profile. Hence, assuming that $\nu_1 < \nu^{(0)} < \nu_2$, and analyzing the principal term of (4.1), we conclude that, depending on the sign of C_{11}, either a crest or a trough of the surface wave may be present over the crest of the bed line. It follows from (3.2) that this sign is determined by the relationship between coefficients β_1 and d_1.

Note 4.2. If $\nu^{(0)} = \nu_n$ corresponds to the eigenvalue of the kernel of the integral equation, we have the particular case mentioned at the beginning of this paper. In fact, for $\nu^{(0)} = \nu_n$ from formulas (1.16) and (1.23) we obtain expression (2.5) which in this particular case links in a linear approximation c and λ_0.

Note 4.3. When $\nu^{(0)} \neq \nu_1$ and the bed line is specified in the form of series (1.3), the analysis of the solution of the linear problem is similar to that presented in Sect. 2, 2.2 with $n = 1$. To allow for subsequent harmonics it is necessary to add to the first harmonic in the first term of series (1.3) the sum of n harmonics of the i th order ($i = 2, 3, \ldots, n$).

If one considers that the solution of the linear problem is determined by the principal term of the complete solution, the result of investigations described in Sect. 2, 2.2 may be also applied to the solution of the nonlinear problem.

5. Existence and uniqueness of solution of the problem. With the use of the Liapunov-Schmidt methods and their developments [7] we establish the following theorems.

Theorem 5.1. For $\nu^{(0)} \neq \nu_n$ the system of Eqs. (1.19), (1.23), (1.24), (1.27), (1.28) and (1.30) has a unique small with respect to ε, and continuous with respect to θ ($0 \leqslant \theta \leqslant 2\pi$) solution $\zeta^*(\theta, \varepsilon)$, $\zeta(\theta, \varepsilon)$, $x(\theta, \varepsilon)/\lambda$, $s(\theta, \varepsilon)$, $A_0(\varepsilon)$,

$\delta'(\varepsilon)$ $(\delta'(\varepsilon) = \delta(\varepsilon) - 1)$, and for small $|\varepsilon| < \varepsilon_0$ this solution is an analytic function of ε.

Theorem 5.2. For $\nu^{(0)} = \nu_1$, where ν_1 is a simple positive eigenvalue, the system of Eqs. (1.19), (1.23), (1.24), (1.27), (1.28) and (1.30) has a unique, small with respect to ε, and continuous with respect to θ $(0 \leqslant \theta \leqslant 2\pi)$ solution $\zeta^*(\theta, \varepsilon)$, $\zeta(\theta, \varepsilon)$, $x(\theta, \varepsilon) / \lambda$, $s(\theta, \varepsilon)$, $A_0(\varepsilon)$, $\delta'(\varepsilon)$, and this solution can be represented in the form of series in $\varepsilon^{1/2}$ convergent for small $|\varepsilon| < \varepsilon_0$.

The proof of these theorems is similar to that given in [8, 9].

These theorems imply the absolute and uniform convergence of series for $\Phi(\theta, \varepsilon)$, $\tau(\theta, \varepsilon)$, $\Phi^*(\theta, \varepsilon)$ and $\tau^*(\theta, \varepsilon)$. The convergence of series in powers of ε and in $\varepsilon^{1/2}$ (for $\nu^{(0)} = \nu_1$) of integrands in (1.9) follows from general theorems of the analysis of substitution of series into series. The convergence of series whose approximate sums are defined by formulas (4.1) and (4.2) is established on the basis of such general theorems.

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